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6.685 Electric Machines 電機機械

課堂講義 2：基本磁路 2005 年 9 月 5 日
Class Notes 2 Magnetic Circuit Basics September 5, 2005

1. 導論

磁路學相似於電路學，提供一種簡單的方法，只要將幾種個別元件組合起來就能用以分析磁場系統。在電路學，這些元件是電源、電阻等，各有不同的電流、電壓值，使用導線連接在一起，其行為則以網路的限制條件（克希荷夫電壓和電流定律[1]）和基本關係例如歐姆定律[2]來描述；在磁路學，合成參數稱為磁阻[3]（磁阻的倒數稱為磁導[4]）。相當於導線的部份則為高磁導的磁路元件。當然高導磁係數就對應於高導電係數。
將磁場系統化成幾個合成參數元件，並使用網路的限制和基本關係，可以簡化分析磁場系統的方法。

1 Introduction

Magnetic Circuits offer, as do electric circuits, a way of simplifying the analysis of magnetic field systems which can be represented as having a collection of discrete elements. In electric circuits the elements are sources, resistors and so forth which are represented as having discrete currents and voltages. These elements are connected together with ‘wires’ and their behavior is described by network constraints (Kirchhoff’s voltage and current laws) and by constitutive relationships such as Ohm’s Law. In magnetic circuits the lumped parameters are called ‘Reluctances’(the inverse of ‘Reluctance’ is called ‘Permeance’). The analog to a ‘wire’ is referred to as a high permeance magnetic circuit element. Of course high permeability is the analog of high conductivity.
By organizing magnetic field systems into lumped parameter elements and using network

[1] 克希荷夫電壓和電流定律 Kirchhoff’s voltage and current laws
[2] 歐姆定律 Ohm’s Law
[3] 磁阻 Reluctance
[4] 磁導 Permeance

constraints and constitutive relationships we can simplify the analysis of such systems.

2. 電路

首先，讓我們複習電路是如何定義的。從兩個守恆定律開始：電荷守恆和法拉第定律，加上適當的簡單假設，可以導出兩個基本的電路限制條件，稱為克希荷夫定律。

2 Electric Circuits

First, let us review how Electric Circuits are defined. We start with two conservation laws: conservation of charge and Faraday's Law. From these we can, with appropriate simplifying assumptions, derive the two fundamental circuit constraints embodied in Kirchhoff's laws.

2.1 克希荷夫電流定律[5]

電荷守恆可以用下列的積分式表示：

2.1 KCL

Conservation of charge could be written in integral form as:

$$\oiint \vec{J} \cdot \vec{n} da + \int_{\text{volume}} \frac{d\rho_f}{dt} dv = 0 \qquad (1)$$

這公式可以簡單敘述為：從某空間流出的電流及該空間內的自由電荷變化率總和為零。
定義單一電流為通過部份表面的電流密度的積分：
This simply states that the sum of current out of some volume of space and rate of change of free charge in that space must be zero.
Now, if we define a discrete current to be the integral of current density crossing through a part of the surface:

$$i_k = -\iint_{\text{surface}_k} \vec{J} \cdot \vec{n} da \qquad (2)$$

並且假設在該容積內沒有電荷累積 (一般電路理論都假設節點[6]的容積非常小，不能累積電荷)，則：
and if we assume that there is no accumulation of charge within the volume (in ordinary circuit theory the nodes are small and do not accumulate charge), we have:

$$\oiint \vec{J} \cdot \vec{n} da = -\sum_k i_k = 0 \qquad (3)$$

將連接到該節點的所有 k 個電流加總，就成立上式，此即為克希荷夫電流定律。

[5] 克希荷夫電流定律 Kirchhoff's Current Law (KCL)
[6] 節點 node

which holds if the sum over the index *k* includes all current paths into the node. This is, of course, KCL.

2.2 克希荷夫電壓定律 Kirchhoff's Voltage Law (KVL)

法拉第定律寫成積分式為：

2.2 KVL

Faraday's Law is, in integral form:

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{n} da \qquad (4)$$

上式等號左邊的封閉積分路徑為等號右邊積分面積的邊緣。
依照一般的方法定義電壓，在元件 *k* 的 *a* 和 *b* 兩點之間：
where the closed loop in the left hand side of the equation is the edge of the surface of the integral on the right hand side.
Now if we define voltage in the usual way, between points *a* and *b* for element *k*:

$$v_k = \int_{a_k}^{b_k} \vec{E} \cdot d\vec{\ell} \qquad (5)$$

假設法拉第定律公式等號右邊（亦即磁感應[7]）為 0，封閉積分公式即成為：
Then, if we assume that the right-hand side of Faraday's Law (that is, magnetic induction) is zero, the loop equation becomes:

$$\sum_k v_k = 0 \qquad (6)$$

由於大部分回路並未含有磁感應，上式乃可成立。但是碰到「接地回路[8]」所引起的電壓時，這往往是非常頭痛的問題。
This works for circuit analysis because most circuits do not involve magnetic induction in the loops. However, it does form the basis for much head scratching over voltages encountered by 'ground loops'.

2.3 基本關係：歐姆定律 Ohm's Law

許多電路使用的材料都具有線性的導電性質，亦即，電場和電流密度的關係為：

2.3 Constitutive Relationship: Ohm's Law

Many of the materials used in electric circuits carry current through a linear conduction mechanism. That is, the relationship between electric field and electric current density is

[7] 磁感應 magnetic induction
[8] 接地回路 ground loop

$$\vec{J} = \sigma \vec{E} \tag{7}$$

假設某物件具有固定截面積，而使電流流過一有限長度，如圖 1 所示。若所流過的電流密度為$\vec{J}$（暫不討論這棒子如何得到此一電流密度）。流過棒子的電流即為：
Suppose, to start, we can identify a piece of stuff which has constant area and which is carrying current over some finite length, as shown in Figure 1. Assume this rod is carrying current density $\vec{J}$ (We won't say anything about how this current density managed to get into the rod, but assume that it is connected to something that can carry current (perhaps a wire....). Total current carried by the rod is simply

$$I = |J|A$$

則此棒兩端之間的電壓為：
And then voltage across the element is:

$$v = \int \vec{E} \cdot d\ell = \frac{\ell}{\sigma A} I$$

從上式，我們總結電阻值為：
from which we conclude the resistance is

$$R = \frac{V}{I} = \frac{\ell}{\sigma A}$$

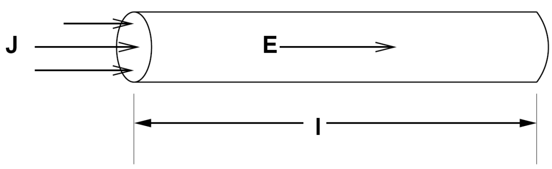

圖 1：簡單棒狀電阻器
Figure 1: Simple Rod Shaped Resistor

即使元件比較複雜，仍可使用合成的參數來代表。例如圖 2 所示的環狀電阻器，這是從某一端看進去的圖，該電阻器具有均勻的截面，深度為 D〈圖上看不到方向〉。假設相較於中間環狀電阻材料，其內外兩部分的元件為非常好的導體。更假設環狀電阻元件的導電係數為σ，而其內外半徑分別為 R_i 和 R_o。
Of course we can still employ the lumped parameter picture even with elements that are more complex. Consider the annular resistor shown in Figure 2. This is an end-on view of something which is uniform in cross-section and has depth D in the direction you can't see. Assume that the inner and outer elements are very good conductors, relative to the annular element in between. Assume further that this element has conductivity σ and inner and outer radii R_i and R_o, respectively.

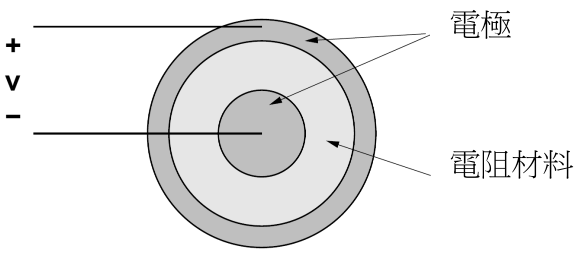

圖 2：環狀電阻器

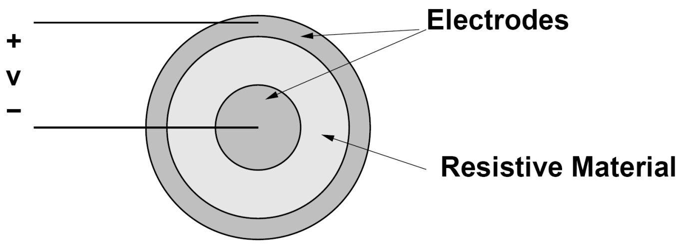

Figure 2: Annular Resistor

如果電流從內電極流向外電極，電流密度將為：
Now, if the thing is carrying current from the inner to the outer electrode, current density would be:

$$\vec{J} = \vec{i}_r J_r(r) = \frac{I}{2\pi D r}$$

電場為：
Electric field is

$$E_r = \frac{J_r}{\sigma} = \frac{I}{2\pi D r \sigma}$$

則電壓為：
Then voltage is

$$v = \int_{R_i}^{R_o} E_r(r) = \frac{I}{2\pi\sigma D} \log \frac{R_o}{R_i}$$

此元件的電阻值即為：
so that we conclude the resistance of this element is

$$R = \frac{\log \frac{R_o}{R_i}}{2\pi\sigma D}$$

3. 磁路類比

在電路的情況，呈現電壓、電流的元件由稱為「導線」的元件連接起來，導線具有零或非常小電壓降，連結的地方稱為「節點」。磁路也有相似的情況：呈現磁動勢和磁通的元件由高導磁係數元件（通常是鐵）連接，此種高磁導元件可比擬於電路的導線。

3 Magnetic Circuit Analogs

In the electric circuit, elements for which voltage and current are defined are connected together by elements thought of as 'wires', or elements with zero or negligible voltage drop. The interconnection points are 'nodes'. In magnetic circuits the analogous thing occurs: elements for which magnetomotive force and flux can be defined are connected together by high permeability magnetic circuit elements (usually iron) which are the analog of wires in electric circuits.

3.1 類比於克希荷夫電流定律

高斯定律[9]：

3.1 Analogy to KCL

Gauss' Law is:

$$\oiint \vec{B} \cdot \vec{n} da = 0 \qquad (8)$$

上式意味從某一空間流出的總通量為零。
定義某一種量，稱之為「磁通」或「磁通管」。可以假想為由許多磁通線捆札在一起。一般而言，磁路元件就是要用來描述磁通線。用數學式表示即為：
which means that the total amount of flux coming out of a region of space is always zero. Now, we will define a quantity which is sometimes called simply 'flux' or a 'flux tube'. This might be thought to be a collection of flux lines that can somehow be bundled together. Generally it is the flux that is identified with a magnetic circuit element. Mathematically it is:

$$\Phi_k = \iint \vec{B} \cdot \vec{n} da \qquad (9)$$

大部分情況下，上式所定義的磁通都在高導磁係數材料所構成的磁路元件內流動，就像電路內的電流在高導電係數材料所作成的「導線」內流動一樣。後面將會證明磁通大部分都被限制在高導磁係數材料內。
In most cases, flux as defined above is carried in magnetic circuit elements which are made of high permeability material, analogous to the 'wires' of high conductivity material which carry current in electric circuits. It is possible to show that flux is largely

[9] 高斯定律 Gauss' Law

contained in such high permeability materials.

從某一空間（節點）流出的所有磁通量相加，其和必為零：
If all of the flux tubes out of some region of space ('node') are considered in the sum, they must add to zero:

$$\sum_k \Phi_k = 0 \tag{10}$$

3.2 類比於克希荷夫電壓定律：磁動勢

安培定律：

3.2 Analogy to KVL: MMF

Ampere's Law is

$$\oint \vec{H} \cdot d\vec{\ell} = \iint \vec{J} \cdot \vec{n} da \tag{11}$$

如同前述的法拉第定律，上式等號左邊的封閉積分路徑為等號右邊積分面積的周圍。類比於「電動勢[10]」（電壓 ），此處定義所謂「磁動勢」：
Where, as for Faraday's Law, the closed contour on the left is the periphery of the (open)surface on the right. Now we define what we call Magnetomotive Force, in direct analog to 'Electromotive Force', (voltage).

$$F_k = \int_{a_k}^{b_k} \vec{H} \cdot d\vec{\ell} \tag{12}$$

另外，定義迴圈所包圍的電流為：
Further, define the current enclosed by a loop to be:

$$F_0 = \iint \vec{J} \cdot \vec{n} da \tag{13}$$

則對應於克希荷夫電壓定律者為：
Then the analogy to KVL is:

$$\sum_k F_k = F_0$$

必須注意的是，此一類比並非完全準確，主要是上式等號右邊有一個「源[11]」的項，而克希荷夫電壓定律則沒有「源」。同時請注意計算所包含的符號。閉迴路積分的正方向（向上）是指面積被包圍在積分路徑左邊時。（這是有名的「右手規則」的另一

[10] 電動勢 Electromotive Force
[11] 源 source

種說法：假如你將右手手指順著積分路徑彎曲，手指指向積分方向，則姆指就指向此面積的正方向）。
Note that the analog is not exact as there is a source term on the right hand side whereas KVL has no source term. Note also that sign counts here. The closed integral is taken in such direction so that the positive sense of the surface enclosed is positive (upwards) when the surface is to the left of the contour.(This is another way of stating the celebrated 'right hand rule': if you wrap your right hand around the contour with your fingers pointing in the direction of the closed contour integration, your thumb is pointing in the positive direction for the surface).

3.3 類比於歐姆定律：磁阻

圖 3 中，兩個高導磁係數物體中間夾著一段間隙。假設兩物體的導磁係數非常高，則可以認為物體當中沒有磁場 H，而其磁動勢或「磁位[12]」基本上是固定值，就如同導線一樣。此處假設間隙 g 很小並且在整個面積 A 內為均勻。而且假設有磁通 Φ 由一物體流向另一物體，磁通量為：

3.3 Analog to Ohm's Law: Reluctance

Consider a 'gap' between two high permeability pieces as shown in Figure 3. If we assume that their permeability is high enough, we can assume that there is no magnetic field H in them and so the MMF or 'magnetic potential' is essentially constant, just like in a wire. For the moment, assume that the gap dimension g is 'small' and uniform over the gap area A. Now, assume that some flux Φ is flowing from one of these to the other. That flux is

$$\Phi = BA$$

此處 B 為間隙內的磁通密度，A 為間隙面積。為了簡化分析，我們忽略了「邊緣」的效應，當然在實際情況邊緣效應必須加以修正。由於自由空間的導磁係數為 μ_0（假設間隙內為自由空間），磁場強度為：
where B is the flux density crossing the gap and A is the gap area. Note that we are ignoring 'fringing' fields in this simplified analysis. This neglect often requires correction in practice. Since the permeability of free space is μ_0, (assuming the gap is indeed filled with 'free space'), magnetic field intensity is

$$H = \frac{B}{\mu_0}$$

間隙內的磁動勢為磁場強度乘以間隙大小。這當然這是假設在間隙大小均勻，而且磁場強度固定的情況：
and gap MMF is just magnetic field intensity times gap dimension. This, of course, assumes that the gap is uniform and that so is the magnetic field intensity:

[12] 磁位 magnetic potential

$$F = \frac{B}{\mu_0} g$$

間隙的磁阻是磁動勢對磁通量的比值：

Which means that the reluctance of the gap is the ratio of MMF to flux:

$$\mathcal{R} = \frac{F}{\Phi} = \frac{g}{\mu_0 A}$$

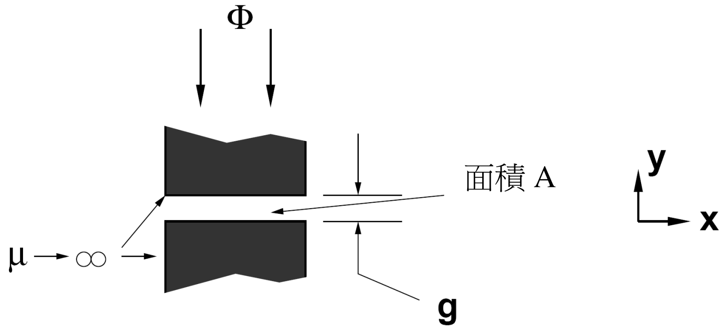

圖 3：氣隙

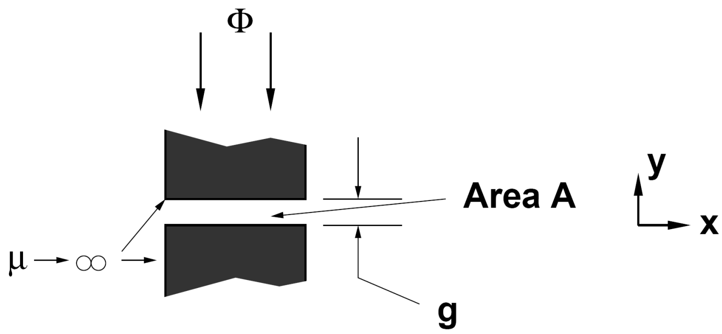

Figure 3: Air Gap

3.4 簡單例

圖 4 所示的磁路由一件高磁導材料構成，其間夾著一段氣隙，而有磁通流過該氣隙。有一個線圈繞過此高磁導材料（此材料一般即稱之為「鐵芯[13]」）。此一裝置的等效電路示於圖 5。

3.4 Simple Case

Consider the magnetic circuit situation shown in Figure 4. Here there is a piece of highly permeable material shaped to carry flux across a single air-gap. A coil is wound through the window in the magnetic material (this shape is usually referred to as a 'core'). The equivalent circuit is shown in Figure 5.

[13] 鐵芯 core

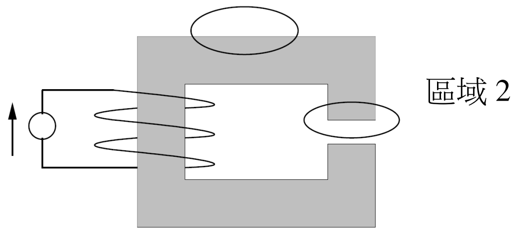

圖 4：單一氣隙的芯

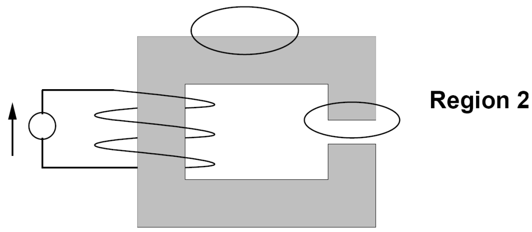

Figure 4: Single air-capped Core

在圖 4 中，如果將此閉迴路的正方向定為繞有線圈的鐵芯部分磁通向上，而在氣隙處磁通向下，則電流在圖上的路徑方向為正。

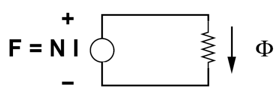

圖 5：等效電路

Note that in Figure 4, if we take as the positive sense of the closed loop a direction which goes vertically upwards through the leg of the core through the coil and then downwards through the gap, the current crosses the surface surrounded by the contour in the positive sense direction.

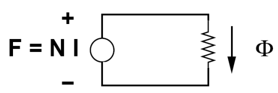

Figure 5: Equivalent Circuit

3.5 磁通侷限

此情況，氣隙的磁阻和前面計算的相同，因此氣隙內的磁通為 $\Phi = \frac{NI}{\mathcal{R}}$。針對圖 4 所示的兩個區域探討一下磁路。首先考慮如圖 6 所示的區域 1。

3.5 Flux Confinement

The gap in this case has the same reluctance as computed earlier, so that the flux in the gap is simply $\Phi = \frac{NI}{\mathcal{R}}$. Now, by focusing on the two regions indicated we might make a few observations about magnetic circuits. First, consider 'region 1' as shown in Figure 6.

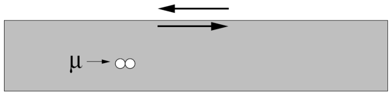

圖 6：磁通侷限之邊界，此為區域 1

Figure 6: Flux Confinement Boundary: This is 'Region 1'

圖中與表面平行的磁場強度 $\vec{H}$ 應該在材料內外都相同。在圖 6 上方邊界所繪的兩個箭頭以及連接的極短垂直線（圖中沒有畫出來）所形成極為細薄的廻圈，做安培定律的計算。如果廻圈內沒有電流奇異點，則環繞此廻圈積分的結果為 0，表示磁場強度在邊界內緣必等於其在邊界外緣。由於 $\vec{B} = \mu\vec{H}$，而且邊界內之高磁導表示其 μ 值非常大，除非 $\vec{B}$ 值特別大，否則 $\vec{H}$ 的值必很小，亦即此磁路的磁場強度 $\vec{H}$ 很小，因此邊界外緣與邊界平行的磁通密度也很小。

In this picture, note that magnetic field $\vec{H}$ parallel to the surface must be the same inside the material as it is outside. Consider Ampere's Law carried out about a very thin loop consisting of the two arrows drawn at the top boundary of the material in Figure 6 with very short vertical paths joining them. If there is no current singularity inside that loop, the integral around it must be zero which means the magnetic field just inside must be the same as the magnetic field outside. Since $\vec{B} = \mu\vec{H}$, and 'highly permeable' means μ is very large, unless $\vec{B}$ is really large, $\vec{H}$ must be quite small. Thus the magnetic circuit has small magnetic field $\vec{H}$ and therefore flux densities parallel to and just outside its boundaries are also small.

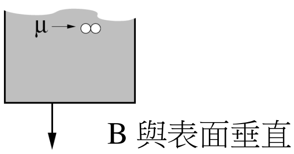

圖 7：氣隙邊界

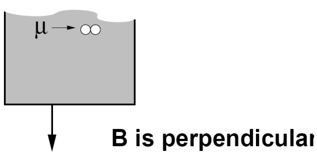

Figure 7: Gap Boundary

由於在磁性材料表面並且和表面平行的磁場強度非常小，從鐵芯內部流出到其表面的磁通，必與表面垂直，如圖 7 所示的氣隙範圍。這情況在區域 1 和區域 2 都正確。但是請注意，讓磁場經過氣隙和在區域 1 產生磁力線者是相同的磁動勢。而在區域 1 流出磁性材料外面的磁力線有非常長的路徑，其數量乃非常少。可知高導磁係數（低磁組）的磁路材料大大的限制了磁通，使大部分磁通線穿過相對高磁導（低磁阻）的氣隙。
At the surface of the magnetic material, since the magnetic field parallel to the surface must be very small, any flux lines that emerge from the core element must be perpendicular to the surface as shown for the gap region in Figure 7. This is true for region 1 as well as for region 2, but note that the total MMF available to drive fields across the gap is the same as would produce field lines from the area of region 1. Since any lines emerging from the magnetic material in region 1 would have very long magnetic paths, they must be very weak. Thus the magnetic circuit material largely confines flux, with only the relatively high permeance (low reluctance) gaps carrying any substantive amount of flux.

3.6 例：C 型鐵芯

圖 8 所示是帶有氣隙的 C 型鐵芯。這是由兩件形狀像字母 C 的高磁導材料所構成，其深度 *D*（與紙面垂直的方向）為均勻，面積 $A=wD$，w 為氣隙的寬度。假設兩個氣隙的面積相等，每一氣隙的磁阻為：

3.6 Example: C-Core

Consider a ‘gapped’ c-core as shown in Figure 8. This is two pieces of highly permeable material shaped generally like ‘C’s. They have uniform depth in the direction you cannot see. We will call that dimension *D*. Of course the area $A = wD$, where w is the width at the gap. We assume the two gaps have the same area. Each of the gaps will have a reluctance

$$\mathcal{R} = \frac{g}{\mu_0 A}$$

若在鐵芯上繞一個 *N* 匝的線圈，如圖 9 所示，並通以電流 *I* 安培，其等效磁路如圖 10 所示。兩個氣隙串聯，並且又和磁動勢源串聯。兩氣隙的磁通相等，其磁動勢相加：
Suppose we wind a coil with *N* turns on this core as shown in Figure 9.Then we put a current *I* in that coil. The magnetic circuit equivalent is shown in Figure 10. The two gaps are in series and, of course, in series with the MMF source. Since the two fluxes are the same and the MMF’s add:

$$F_0 = NI = F_1 + F_2 = 2\mathcal{R}\Phi$$

然後，
And then

$$\Phi = \frac{NI}{2\mathcal{R}} = \frac{\mu_0 ANI}{2g}$$

對應的氣隙磁通密度乃為：
and corresponding flux density in the gaps would be:

$$B_y = \frac{\mu_0 NI}{2g}$$

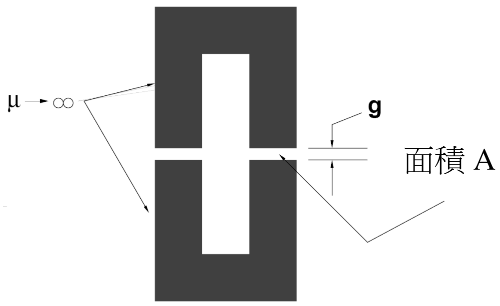

圖 8：帶有氣隙的鐵芯

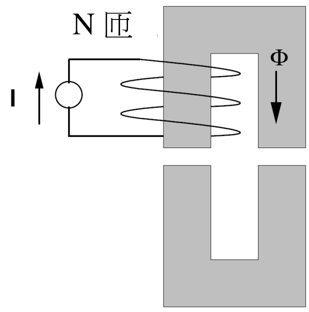

圖 9：帶有線圈和氣隙的鐵芯

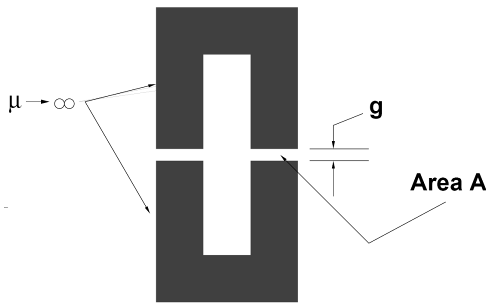

Figure 8: Gapped Core

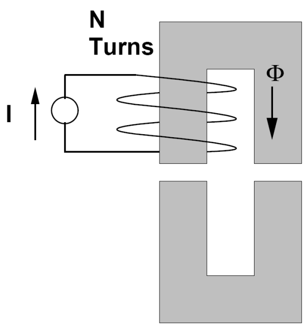

Figure 9: Wound, Gapped Core

3.7 例：具有不同氣隙的鐵芯

第二個例子，考慮像圖 11 這種形狀比較奇怪的鐵芯。假設右邊的氣隙面積是左邊氣隙的兩倍。則有兩個不同的氣隙磁阻：

3.7 Example: Core with Different Gaps

As a second example, consider the perhaps oddly shaped core shown in Figure 11. Suppose the gap on the right has twice the area as the gap on the left. We would have two gap reluctances:

$$\mathcal{R}_1 = \frac{g}{\mu_0 A} \qquad \mathcal{R}_2 = \frac{g}{2\mu_0 A}$$

由於兩個氣隙串聯，磁通相同，總磁阻乃為：
Since the two gaps are in series the flux is the same and the total reluctance is

$$\mathcal{R} = \frac{3}{2}\frac{g}{\mu_0 A}$$

磁路內的磁通為：
Flux in the magnetic circuit loop is

$$\Phi = \frac{F}{\mathcal{R}} = \frac{2}{3}\frac{\mu_0 ANI}{g}$$

流過（例如左邊氣隙）的磁通密度為：
and the flux density across, say, the left hand gap would be:

$$B_y = \frac{\Phi}{A} = \frac{2}{3}\frac{\mu_0 NI}{g}$$

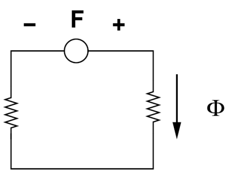

圖 10：等效磁路

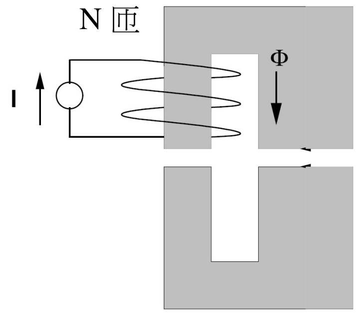

圖 11：有線圈和氣隙的鐵芯，氣隙尺寸不等

$$B_y = \frac{\Phi}{A} = \frac{2}{3}\frac{\mu_0 NI}{g}$$

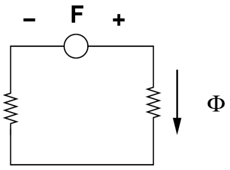

Figure 10: Equivalent Magnetic Circuit

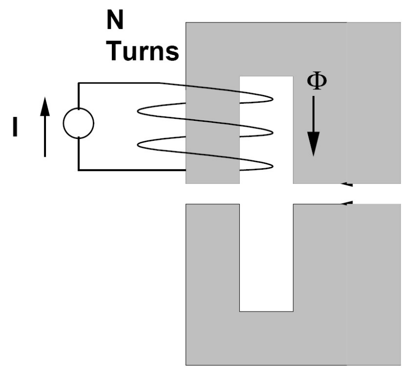

Figure 11: Wound, Gapped Core: Different Gaps